J. Fluid Mech. (1970), *vol.* 40, *part* 4, *pp.* 807–819
Printed in Great Britain

Friction reduction in turbulent flow of polymer solution

By F. A. SEYER

Department of Chemical and Petroleum Engineering, University of Alberta, Edmonton, Alberta, Canada

(Received 15 April 1969)

Recent theoretical results for the turbulent flow of polymer solutions in round tubes have been extended to deduce the similarity laws for the boundary-layer flow of drag-reducing polymer solutions. The analysis shows directly how the drag reduction depends on the elastic properties of the fluid and thereby defines the levels of elasticity necessary to achieve significant reductions in drag.

Calculations employing available physical property measurements of highly elastic (0·1 %) and moderately elastic (0·01 %) polymer solutions indicate that, for boundary layers on large objects, drag reduction may not occur at polymer concentrations that are economically attractive. For example, at a Reynolds number of 10^9 the reduction in drag is predicted to be 60 % and 10 % for the concentrated and dilute polymer solutions respectively. Some savings in polymer however, may be realized by special injection techniques or fluid systems with specially tailored properties.

1. Introduction

The phenomena of turbulent flow-drag reduction has received considerable attention owing to the many applications of both theoretical and pragmatic interest. Drag reduction in round tubes as effected by dilute polymer solutions has by now been well documented (Fabula 1965; Metzner & Park 1964; Seyer & Metzner 1967; Virk *et al.* 1967) and a recent review is available (Patterson, Zakin & Rodriguez 1969) which summarizes typical data and discusses empirical and theoretical analyses that have appeared. Little work, however, has appeared which is related to the reduction in drag of turbulent boundary-layer flows.

Experiments have shown (Vogel & Patterson 1964; Lang & Patrick 1966) that significant reductions in drag can occur when bluff objects move through dilute polymer solutions, although in at least one instance an augmentation of the drag has been observed (Merrill, Smith & Chung 1966). Similarly, drag reduction has been observed on addition of polymeric additives to the boundary layer of streamlined objects (Crawford & Pruitt 1963; Vogel & Patterson 1964). An analysis by Granville (1967) is available in which the similarity laws for boundary flows of dilute polymer solutions are deduced using dimensional arguments. In this analysis, which lacks a physical interpretation of the drag reduction, the arbitrary functions, as well as their dependence on any of the material parameters of the fluids, must be determined empirically.

In earlier studies one of the primary obstacles to formulating a quantitative understanding of the drag reduction has been the difficulty in applying conventional measurement techniques to these systems (Astarita & Nicodemo 1966; Fabula 1966; Metzner & Astarita 1967; Metzner & Seyer 1966; Savins 1965; Wetzel & Tsai 1968). Additionally it is noted that most of the empirical studies lack an adequate definition of the pertinent physical properties of the polymer solutions, and therefore offer little hope of being applicable to systems other than those from which the correlations were derived. Recently, however (Seyer & Metzner 1969) (based on tube flow experiments in which an optical technique was used to study the turbulence) it has been possible to deduce the similarity laws and their detailed dependence on the relaxation time of the fluid. It is the purpose of this paper to extend these results in formulating the similarity laws for drag reduction in a boundary-layer system, and to consider in detail the influence of physical properties on the drag reduction at Reynolds numbers which are sufficiently large to be of practical importance.

2. Analysis

In the flow of Newtonian fluids the extensive data of Laufer (1954) and Klebanoff (1955), among others (Hinze 1959) serve to illustrate the importance of the flow in the wall region in governing the macroscopic features of the flow field. In drag-reducing systems the importance of the wall region has been demonstrated experimentally by the recent work of Wells & Spangler (1967) in which large reductions in drag were obtained by injecting the polymeric additive within the boundary-layer region of a tube.

In the region of the viscous sublayer the visual studies of Runstaldler, Kline & Reynolds (1962), and Corino & Brodkey (1969) and the detailed velocity measurements of Bakewell & Lumely (1967) serve in part to define the mechanics of Reynolds stresses. Qualitatively, in the region of the wall extending to approximately $y^+ = 30$, the turbulent momentum transfer rates, in a direction normal to the wall, are seen to be governed primarily by large eddies that occur at random locations along the wall. The large eddies initially appear as concentrations of low momentum fluid which subsequently eject a mass of fluid toward the main stream where the identity of this fluid is rapidly lost. Corino & Brodkey show that as much as 70 % of the turbulent Reynolds stress can be accounted for by consideration of this flow. Bakewell's velocity measurements show that the eddies occur as counter-rotating pairs with axes of rotation along the direction of mean flow and that ejection of the low momentum fluid may be identified with the radially directed flow between the two eddies. This picture of the eddy structure is in qualitative agreement with an earlier but highly simplified model of wall eddies considered by Townsend (1956).

In the ejection process, analysis of Bakewell's measurement has shown (Seyer & Metzner 1969) that the flow process may be approximated by a pure elongational flow field much like that observed, for example, in converging flows from a large to a smaller duct. It is well known from both experimental and theoretical analyses (Astarita 1967, 1968; Ballman 1965; Marshall & Metzner 1967; Metzner

& Metzner 1969) that viscoelastic materials may offer a huge resistance to elongational flow depending on the relative magnitude of the rate of elongation and a characteristic time scale of the fluid. For example, if a Maxwell model is selected to portray the properties of a sheet of material (Metzner 1967) it is easily shown that the elongational viscosity, defined as the longitudinal stress divided by the rate of elongation is given by

$$\mu_e = \frac{4\mu}{1-(2\theta\Gamma)^2},$$

in which μ is the ordinary shearing viscosity, θ the relaxation time of the fluid and Γ the rate of elongation. Thus for fluids with a finite relaxation time, the elongational viscosity is seen to always exceed the ordinary shearing viscosity and in fact is predicted to increase without limit as $2\theta\Gamma$ approaches unity. Of significance is the fact that Metzner & Metzner (1969) have measured elongational viscosities of the order of 10,000 times the shearing viscosity for fluids commonly employed as drag reducers.

For a turbulent flow field the stresses which sustain the elongational deformations normal to the wall are primarily inertial in nature and therefore, at a given Reynolds number, will be equivalent in Newtonian and drag-reducing systems. As the elongational viscosity given in the preceding equation must be larger than for Newtonian fluids, the magnitude depending on the relaxation time, it follows that the rate of elongation must be lowered. In the model developed by Seyer & Metzner (1969) the rate of elongation, Γ, is shown to be directly related to the local rate of momentum transfer so that a reduction in Γ implies directly that a reduced drag must occur.

The preceding arguments while not lending themselves directly to a quantitative calculation of the amount of drag reduction, show the importance of a fluid relaxation time in governing the turbulence processes in the wall region. It has been pointed out in a number of instances (Astarita 1966, 1967; Truesdell 1964) that characterization of viscoelastic materials involves consideration of at least one material parameter having the dimensions of time. Therefore the following dimensional arguments (up to and including equation (18)), which depend primarily on introduction of a scalar having the dimension of time to characterize the material, may be developed without reference to an assumed constitutive equation.

Velocity profile

Considering a turbulent boundary layer on a flat plate with U_∞ the free-stream velocity and y the distance perpendicular to the plate, it is assumed (following Millikan 1939, p. 386) that in the wall region the axial velocity is given by:

$$u = f_2(\tau_w, y, \rho, \mu, \theta), \tag{1}$$

in which μ, ρ, and θ are the viscosity, density and a relaxation time of the material respectively. For the polymer solutions of interest the viscosity and relaxation time are in general dependent on the rate of deformation or equivalently shearing stress. Thus in (1) μ and θ are determined by the local value of the wall shear stress, τ_w.

Velocity profile and turbulence intensity measurements in the turbulent core of a round tube, obtained using an optical technique (Seyer & Metzner 1969), have shown that there is little measurable difference in the core turbulence from that of Newtonian fluids. Similar results for the velocity profile in the core region have been obtained by Nicodemo, Acierno & Astarita (1969) using a specially calibrated pitot tube. Therefore in the outer region of the boundary layer of thickness δ, the velocity defect is assumed to apply so that

$$U_\infty - u = f_3(\tau_w, y, \delta, \rho). \tag{2}$$

Thus over the entire boundary layer

$$u = f_1(\tau_w, y, \rho, \delta, \mu, \theta). \tag{3}$$

With the dimensionless groups

$$\left.\begin{aligned} Z &= \delta u^*/\nu, \\ \zeta &= y/\delta, \\ \theta^+ &= \theta(u^*)^2/\nu, \end{aligned}\right\} \tag{4}$$

and

in which $u^* = \sqrt{(\tau_w/\rho)}$ and $\nu = \mu/\rho$, equations (1), (2) and (3) are of the form

$$\left.\begin{aligned} u/u^* &= f_1(Z, \zeta, \theta^+), \\ U_\infty/u^* &= F(Z, \theta^+), \end{aligned}\right\} \tag{5}$$

or

$$u/u^* = f_2(Z\zeta, \theta^+), \tag{6}$$

$$(U_\infty - u)/u^* = f_3(\zeta). \tag{7}$$

Assuming a region of overlap for the velocities given by (6) and (7) so that with equation (5) we have

$$-\zeta \frac{\partial f_3(\zeta)}{\partial \zeta} = Z \frac{\partial F(Z, \theta^+)}{\partial Z} = A$$

in which A must be a constant independent of the dimensionless groups. Integration results in

$$U_\infty/u^* = A \ln Z + B(\theta^+), \tag{8}$$

and

$$(U_\infty - u)/u^* = -A \ln \zeta + C. \tag{9}$$

Equation (8) is independent of position whereas (9) is strictly valid only in the region of overlap. For Newtonian fluids the equation (9) is allowed to empirically describe the velocity over the entire boundary layer by allowing C to become a function of position, $C'(\zeta)$ say. Thus combining (8) and (9) we have, on introducing the dimensionless velocity $u^+ = u/u^*$,

$$u^+ = A \ln y^+ + B(\theta^+) - C'(\zeta),$$

which is valid over the entire boundary layer from the region of overlap to the outer edge of $y = \delta$. Generally, available velocity profile data (Bogue & Metzner 1963; Hinze 1959; Seyer & Metzner 1969) show that the function C' is the minor term in the preceding relation and for practical purposes may be safely neglected. Thus we have

$$u^+ = A \ln y^+ + B(\theta^+), \tag{10}$$

which is a statement of the velocity law and is approximately valid over the boundary layer from the region of overlap to the outer edge at $y = \delta$.

In (10) the intercept is an arbitrary but universal function of the dimensionless group θ^+ and A is a constant which is predicted to be the same as for Newtonian fluids. Thus the effect of elasticity is to change the intercept in (10) through changes in the group θ^+. A similar result has been proposed earlier by Granville (1967) in which, however, it is necessary to empirically determine the dependence of B on the elastic properties plus several other dimensionless groups which are assumed necessary to portray the physical properties of the polymer solution.

Drag coefficient

The drag on the plate, up to a position x along the plate, is given by the difference in momentum flux between the leading edge and that of position x along the plate. For Newtonian fluids, if contributions from the turbulent normal Reynolds stresses are ignored, the drag $D(x)$ on a plate of unit width up to a position x is given by Schlichting (1968)

$$D(x) = \int_0^\delta u(U_\infty - u)\,dy, \tag{11}$$

$$\tau_w = \rho \frac{d}{dx}\int_0^\delta u(U_\infty - u)\,dy, \tag{12}$$

in which τ_w denotes the local value of the wall shearing stress. For the elastic systems of interest herein little can be said concerning the magnitude of the Reynolds stresses although, as noted before, measurements of turbulence intensity in the elastic systems suggest that the turbulence in the logarithmic region does not differ measurably from that of Newtonian fluids. Thus as a first approximation (11) and (12) are assumed to also portray the major effects in elastic systems.

Substituting (10) into (11) and integrating:

$$\left.\begin{aligned} &D(x) = \rho A u^*(U_\infty - 2Au^*)\,\delta, \\ \text{or}\quad &D(x) = \rho A \nu (U_\infty - 2Au^*) \exp\left[\frac{1}{A}\left(\frac{U_\infty}{u^*} - B\right)\right] \end{aligned}\right\} \tag{13}$$

on substituting for δ from equation (8). Following von Karman (1934) the substitutions

$$\Psi' = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} \tag{14}$$

and

$$\sigma = \sqrt{(2/\Psi')}, \tag{15}$$

where Ψ' is the local value of the dimensionless drag coefficient, in equation (13) result in

$$D(x) = A\rho\nu U_\infty \left(1 - \frac{2A}{\sigma}\right) \exp\left(\frac{\sigma - B}{A}\right). \tag{16}$$

Defining the local Reynolds number as

$$N_{R,x} = \frac{U_\infty x}{\nu} \tag{17}$$

and differentiating (16)

$$\frac{1}{\rho U_\infty^2}\frac{dD(x)}{dx} = A\frac{d}{dN_{R,x}}\left[\left(1-\frac{2A}{\sigma}\right)\exp\left(\frac{\sigma-B}{A}\right)\right]$$

or rearranging, and integrating from the leading edge of the plate

$$N_{R,x} = A\int_0^\sigma \sigma^2 d\left[\left(1-\frac{2A}{\sigma}\right)\exp\left(\frac{\sigma-B}{A}\right)\right]. \tag{18}$$

Limiting cases: highly elastic system

To integrate (18) it is necessary to specify the function B and its dependence on σ or, equivalently, the local shearing stress. The function B as determined by Seyer & Metzner (1969) from pipe flow measurements is shown in figure 1. It

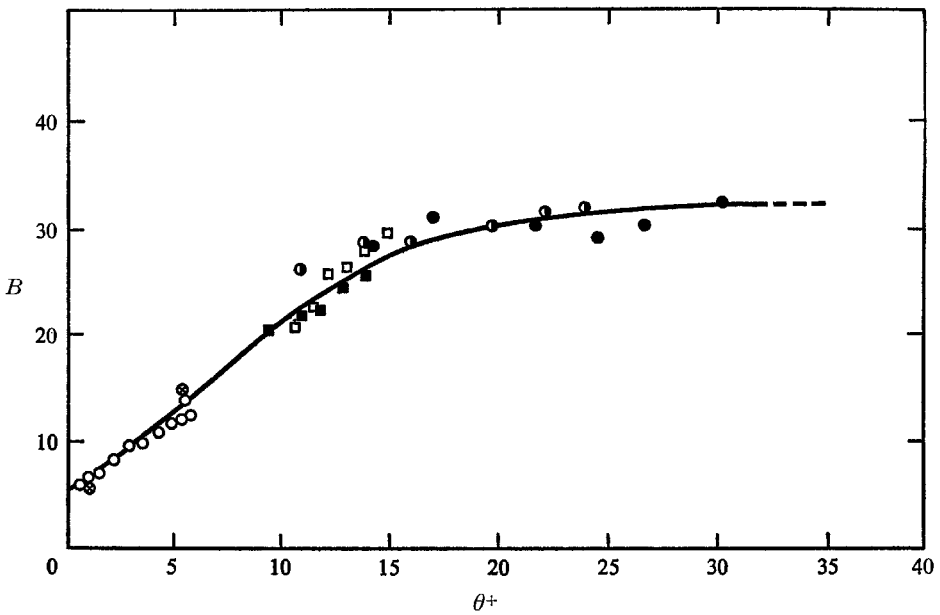

FIGURE 1. Intercept function for similarity laws.

Tube	Evaluated from velocity profile 0·1 %	Evaluated from drag coefficient 0·1 %	0·2 %	0·4 %
1 in.	⊗	○	◑	●
2 in.	—	—	□	■

should be noted that the data in figure 1 are based on a limited range of pipe diameters and a single polymeric additive and therefore the correlation has not been severely tested. They do, however, encompass a sufficiently broad range of the primary variables involved, that is shearing stress and relaxation time, to provide a basis for integration of (18). While other pipe flow data are available from which B can be estimated and therefore compared with the function in

figure 1, the absence of detailed physical property measurements makes a detailed comparison impossible.

The data in figure 1 have been derived primarily from frictional measurements. However, the close agreement between the two values obtained from velocity profiles, using a photographic technique, and the frictional measurements is evident. Values of the relaxation time used in determining θ^+ have been estimated from normal stress measurements by assuming a Maxwell model to portray the fluid properties (Seyer 1968). While this model is not correct in detail it is known to portray adequately the major effects observed in viscoelastic systems in a variety of flows (Etter & Schowalter 1965; Marshall & Metzner 1967; Meister & Biggs 1969; Seyer & Metzner 1969). In particular, calculations presented by Seyer & Metzner relative to the large eddy structure discussed earlier show, as expected from the equation for elongational viscosity, that drag reduction occurred when the group $2\theta\Gamma$ became significant compared to unity. Thus, although the intercept function B determined by Seyer & Metzner depends on the assumed Maxwell model, this does not appear to be a serious limitation.

In the limit for vanishingly small elasticity, that is $\theta^+ \to 0$, $B(0) = 5{\cdot}6$ (Bogue & Metzner 1963). Practically the data indicate that little effect of elasticity will occur for θ^+ less than approximately unity. For $\theta^+ > 20$, the function becomes approximately constant and thus for those systems which are sufficiently elastic B will be constant over the length of the plate

$$B(\theta^+ > 20) = \hat{B} \cong 32,$$

which represents the maximum drag reduction that can occur for the plate. Following von Karman (1934) and integrating (18) we find

$$N_{R,x} = A\sigma^2 \exp\left[\frac{\sigma - \hat{B}}{A}\right]\left[1 - 4\frac{A}{\sigma} + 6\frac{A^2}{\sigma^2} - 6\frac{A^2}{\sigma^2 \exp(\alpha/A)}\right]. \tag{19}$$

For large Reynolds numbers, neglecting terms in $1/\sigma$ and higher powers

$$N_{R,x} \simeq A\sigma^2 \exp\left(\frac{\sigma - \hat{B}}{A}\right). \tag{20}$$

It is noted that neglect of the higher-order terms in (19) is in the case of drag-reducing systems, a better approximation than for Newtonian fluids owing to the increased values of σ that must occur at a given Reynolds number. For a plate of unit width and length L, the average drag coefficient is defined as

$$\begin{aligned} C_f = \frac{D(x)}{\frac{1}{2}\rho U_\infty^2 L} &= \frac{A(1 - 2A/\sigma)\exp\{(\sigma - \hat{B})/A\}}{\frac{1}{2}N_{R,L}} \\ &= \frac{2\{1 - 2A/\sigma\}}{\sigma^2(1 - (4A/\sigma) + (6A^2/\sigma^2) - \ldots)} \\ &\simeq 2/\sigma^2, \end{aligned} \tag{21}$$

for large Reynolds number. Therefore using (21) in (20) and rearranging

$$\sqrt{(2/C_f)} = A \ln(N_R C_f) - A \ln 2A + \hat{B}, \tag{22}$$

which is the desired relationship. Equation (22) is identical in form to previous results for Newtonian fluids but with a modified value for the intercept. As the intercept is significantly larger than for Newtonian fluids, equation (22) requires that for a given Reynolds number a reduction in C_f from the Newtonian value must occur.

Moderate elasticity

For very dilute polymer solutions, say in the order of 0·01 % by weight or less reference to earlier data (Seyer & Metzner 1969) indicates that, except for a very small region near the leading edge of the plate, the group θ^+ is confined to values much less than 20. For example, considering figure 1 at $\theta^+ = 20$, the corresponding value of shearing stress for the 0·01 % solution studied by Seyer is approximately 10 psf. For a Newtonian fluid at $U_\infty = 30$ ft./sec a straightforward calculation (Schlichting 1968) indicates that shearing stresses of 10 psf or greater only occur over approximately the first inch of the plate. It is therefore clear that these values of θ^+ can contribute little to modifying the overall drag coefficient and accordingly it is assumed that B is a linear function of the group θ^+. In this region

$$B(\theta^+ < 20) = B' = \alpha + \beta'\theta^+, \tag{23}$$

in which an adequate fit is obtained with

$$\alpha = 5{\cdot}6,$$

$$\beta' = 1{\cdot}55.$$

Noting in (4) that the group θ^+ is a function of u^*, (23) must be modified before (18) can be integrated. Earlier data (Oliver 1966) which can be used to estimate $\theta(\tau_w)$ for the dilute solutions indicate that over the range of shearing stress of interest

$$\theta^+ \propto \sqrt{\tau_w},$$

which in (23) allows us to write

$$B' = \alpha + \beta u^* \quad (\theta^+ < 20), \tag{24}$$

$$B' = \alpha + \beta(U_\infty/\sigma), \tag{25}$$

in which β is a constant characteristic of the elasticity of a given polymer solution, and can be determined from a plot of θ^+ *versus* u^*.

Substituting (25) into (18)

$$N_{R,x} = A\exp(-\alpha/A)\int_0^\sigma (\sigma^2 - 2A\sigma)\exp\left(\frac{\sigma-\beta U_\infty/\sigma}{A}\right)\left(\frac{1+\beta U_\infty/\sigma^2}{A}\right)d\sigma$$
$$+ 2A^2\exp(-\alpha/A)\int_0^\sigma \exp\left(\frac{\sigma-\beta U_\infty/\sigma}{A}\right)d\sigma. \tag{26}$$

Equation (26) is now expanded and integrated by parts, each integral yielding a power series in $1/\sigma$. With a significant amount of algebra and combining all of the terms, there results

$$N_{R,x} = A\sigma^2\exp\left(\frac{\sigma-B'}{A}\right)\left[1+O\left(\frac{1}{\sigma}\right)\right], \tag{27}$$

in which terms of order $1/\sigma$ and higher are assumed negligible compared to unity. With the definition of the drag coefficient used in (21)

$$C_f = \frac{2A\left(1-\frac{2A}{\sigma}\right)\exp\left(\frac{\sigma-B'}{A}\right)}{A\sigma^2\exp\left(\frac{\sigma-B'}{A}\right)}$$
$$\simeq 2/\sigma^2.$$

Therefore, for large Reynolds number, (27) may be written

$$\sqrt{(2/C_f)} = A\ln N_R\, C_f - A\ln 2A + B', \tag{28}$$

in which B' is given by either (23) or (25). Equation (28) shows that for the region over which B is a linear function of the same form of relationship for the drag coefficient is applicable (cf. (22)); in this case, however, the intercept is dependent on the prevailing value of the drag coefficient, or equivalently, the shearing stress. As an alternate approximation, in common with a number of other studies (Astarita, Greco & Nicodemo 1969; Elata, Leher & Kahanovitz 1966), one may assume a shearing independent relaxation time in estimating θ^+. In this case θ^+ becomes directly proportional to shearing stress; however, the essential form of (28) is retained.

3. Results and discussion

In figure 2 the drag coefficient has been plotted as a function of length Reynolds number for maximum drag reduction. Included for comparison purposes is the curve for Newtonian fluids, calculated using equation (22) with $B = 5{\cdot}6$ and $A = 2{\cdot}5$ (Schlichting 1968).

At a Reynolds number of approximately 2×10^6 the calculations predict that the curve of maximum drag reduction will intersect with the curve for laminar flow. Reference to earlier data for flow in pipes (Seyer & Metzner 1969) indicates, however, that the behaviour in this region will be more nearly as indicated by the dashed portion of the curve. That is, a gradual departure from the laminar relation occurs which is followed by a very broad region of transition to fully developed turbulence. This idea is supported qualitatively by the recent observations of Tanner (private communication) in which visual studies have been made of the growth of turbulent spots in drag reducing systems. At a given Reynolds number the growth angle of a turbulent spot was seen to be markedly reduced from that of a Newtonian fluid implying that transitional flow exists over a broad range of Reynolds number based on length. For the large Reynolds numbers of interest, however, say a large ship at 50 ft./sec and 600 ft. long, the broadened transition region would still be confined to a very short portion of the leading edge.

At a Reynolds number of 10^9 the curves indicate that the maximum percentage reduction in skin drag from the value for Newtonian fluids is 60 % of the Newtonian value. While this is clearly a significant reduction in skin drag it is

necessary to comment under what concentrations of polymeric additive this reduction might be achieved.

In figure 3 the dimensionless group θ^+ is given as a function of shearing stress for two concentrations of the same polymeric additive in water. The calculated curves are based on earlier normal stress data obtained by Oliver (1966) and by

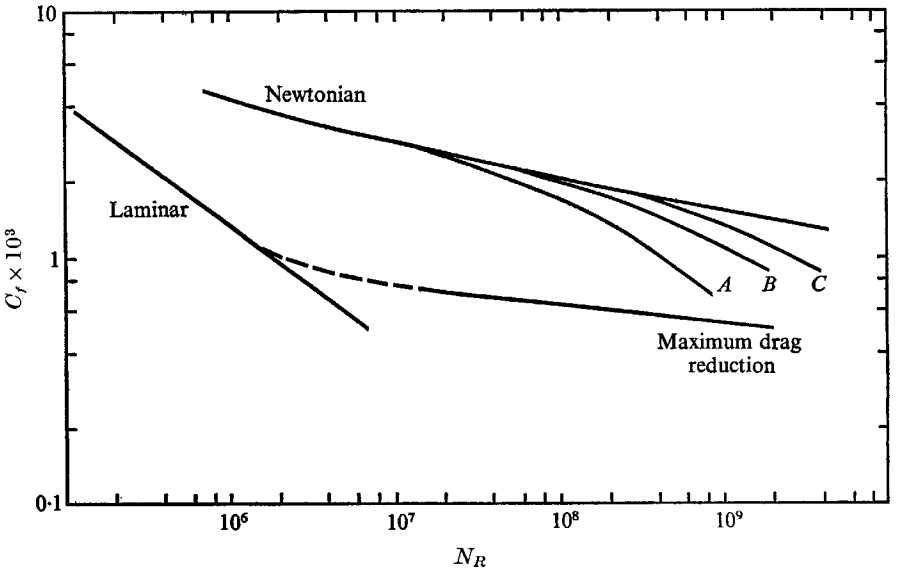

FIGURE 2. Predicted drag coefficients. Curve: A, $l = 100$ ft.; B, $l = 500$ ft.; C, $l = 1000$ ft.

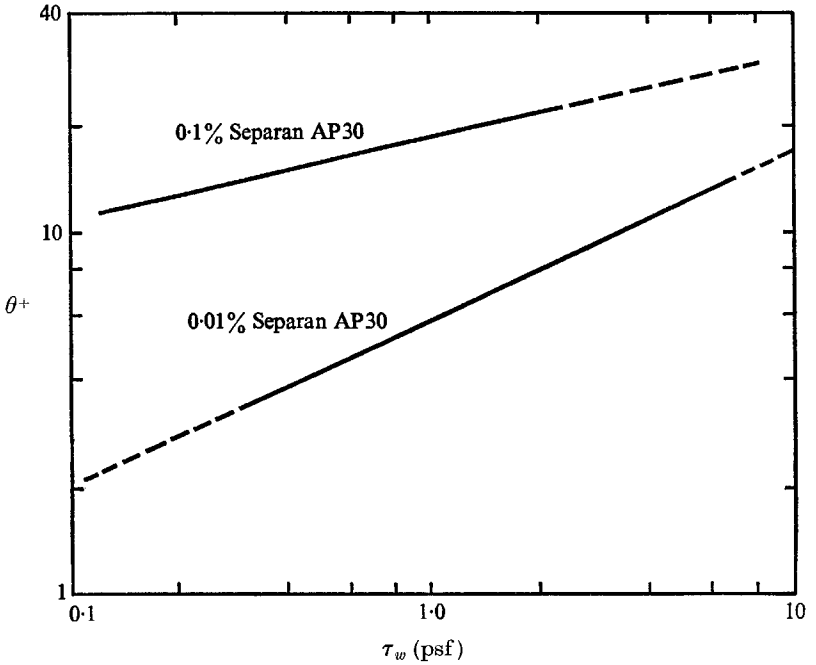

FIGURE 3. Dimensionless relaxation time for 0·1 % and 0·01 % polymer solutions.

Seyer & Metzner (1967) for water solutions of Separan AP 30. Considering the curve for the 0·1 % solution, the parameter θ^+ will exceed 20, providing the shearing stress is greater than about unity. For a boundary-layer flow with $L = 600$ ft. and $U_\infty = 50$ ft./sec figure 2 predicts that at the end of the plate $\tau_w \simeq 1{\cdot}25$ psf and therefore that this material will behave as a maximum drag reducer.

For the 0·01 % solution a similar calculation shows that except for a very small strip less than 1 ft. in extent, near the leading edge of the plate, $\theta^+ < 20$ and therefore the linear approximation B' may be used.

From the curve in figure 3 for the 0·01 % solution, equation (25) becomes

$$B' \simeq 5{\cdot}6 + 12{\cdot}1(U_\infty/\sigma).$$

Calculated values of the drag coefficient for the 0·01 % solution have been included in figure 2 for several plate lengths. In this case the drag reduction becomes a function of plate length because in (28) the drag coefficient is not a unique function of the Reynolds number. For example, at a Reynolds number of 10^9 the reductions in drag are roughly 59 %, 23 % and 10 % of the Newtonian value for the 100, 500 and 1000 ft. plates respectively. Qualitatively the approach to Newtonian behaviour with increasing plate length results from the generally smaller values of shearing stress and correspondingly smaller values of the dimensionless relaxation time θ^+ that prevail for longer plates. This situation is similar to the decrease in drag reduction that occurs with increasing tube diameter.

Considering a large ship, say in the order of 1000 ft. in length at a speed of 50 ft./sec the calculations illustrate that essentially no drag reduction can occur if the boundary-layer concentration is maintained at 0·01 % by weight of the polymer considered in this work. Although at Reynolds numbers of practical significance drag reduction will occur with the concentrated solution, the rather large concentration involved may be prohibitive economically. Theoretically the analysis in this paper indicates that the necessary concentration of polymer need only be maintained within the inner region of the boundary layer, defined by $y^+ < 30$ say, rather than throughout the entire boundary layer. This observation is of course supported by the experimental work of Wells & Spangler (1967) noted earlier. Thus to minimize the polymer consumption it is clear that small amounts of the polymer should be carefully injected at frequent intervals along the boundary layer such that the desired concentration of polymer would be confined primarily to a thin region very near the wall. The alternative, that is injecting all of the polymer at the leading edge of the vehicle, would require that the entire boundary layer (to $y = \delta$) be maintained at the desired concentration level.

Noting from figure 3 that the effectiveness of the polymer as a drag reducer generally decreases along the plate, owing to the decrease of θ^+ with decreasing shearing stress, it is possible that significant savings of polymer could be effected by a 'programmed' injection. For example, over the first half of the plate, where the shearing stress is high a 0·01 % solution might be employed, while over the remainder of the plate a higher concentration would be necessary to maintain

the parameter θ^+ at a value high enough to effect significant drag reduction. In this context the recent drag reduction studies (Savins 1967, 1968*a*, 1968*b*; White 1967), in which micellar systems have been employed to reduce the drag should be noted. In particular White's data indicate that contrary to the behaviour of polymer solutions the drag reduction can increase with increasing pipe diameter. Qualitatively, referring to figure 3, this effect is predicted to occur with polymer solutions if the $\tau_w - \theta^+$ curves were to have a negative slope or, equivalently, if the effect of elasticity *increases* with decreasing shearing stress. In the case, for example, of boundary-layer flows on large ships, this effect is particularly attractive owing to the modest levels of shearing stress that prevail along most of the boundary layer.

4. Summarizing comments

The similarity laws for the turbulent boundary-layer flow of drag reducing polymer solutions have been formulated. Owing to the decrease in elastic effects with decreasing shear stress (at least for materials considered herein for which property measurements are available) the effectiveness of the drag reducer decreases along a plate or, equivalently, with plate length. This observation indicates that further studies using the micellar systems noted earlier or with other materials having specially tailored properties is warranted. The major effects that are predicted to occur suggest study is warranted in verifying the quantitative details of the function B.

The recent observations of Tanner (private communication) for flow over a flat plate and the earlier turbulence measurements in round tubes (Seyer & Metzner 1969) indicate that the concentrated polymer solutions have a major influence on the transition to turbulence. If the transition to turbulence can be substantially delayed, or the intensity of the large eddies that exist near the wall significantly reduced, then the reduction in turbulent transport of polymer away from the wall region may be a major factor in determining the total polymer consumption. Conceivably, to optimize this effect a highly concentrated polymer solution might be most effective. Further studies are necessary, however, in order to quantitatively describe the delay in transition as well as any decrease in turbulent diffusivity near the wall.

REFERENCES

Astarita, G. 1966 *Can. J. Chem. Engng*, **44**, 59.

Astarita, G. 1967 *Ind. Eng. Chem. Fund.* **6**, 257.

Astarita, G. 1968 *Ind. Eng. Chem. Fund.* **7**, 171.

Astarita, G., Greco, G. & Nicodemo, L. 1969 *Am. Ind. Chem. J.* **15**, 564.

Astarita, G. & Nicodemo, L. 1966 *Am. Ind. Chem. J.* **12**, 478.

Bakewell, H. P. & Lumely, J. L. 1967 *Phys. Fluids*, **10**, 1880.

Ballman, R. L. 1965 *Rheologica Acta*, **4**, 137.

Bogue, D. C. & Metzner, A. B. 1963 *Ind. Engng Chem. Fund.* **2**, 143.

Corino, E. R. & Brodkey, R. S. 1969 *J. Fluid Mech.* **37**, 1.

Crawford, H. R. & Pruitt, G. T. 1963 Paper presented at *Symp. on Non-Newtonian Fluid Mech.* 56th Annual *A.I.Ch.E.* meeting.

ELATA, A., LEHRER, J. & KAHANOVITZ, A. 1966 *Israel J. Tech.* **4**, 87.

ETTER, I. & SCHOWALTER, W. R. 1965 *Trans. Soc. Rheol.* **9**, 351.

FABULA, A. G. 1965 *Proc. Fourth Int. Cong. on Rheology*, part 3. (Ed. E. H. Lee.) New York: Interscience.

FABULA, A. G. 1966 *Proc. Sixth Naval Hydrodynamics Symposium*. Washington, D.C.: Office Naval Research.

GRANVILLE, P. S. 1967 *Hydromechanics Laboratory Report* 2502. Washington, D.C.: Naval Ship Research and Dev. Center.

HINZE, J. O. 1959 *Turbulence*. New York: McGraw-Hill.

KLEBANOFF, P. S. 1955 *N.A.C.A. Rep.* 1247.

LANG, T. G. & PATRICK, H. V. L. 1966 Paper presented at *Annual meeting of A.S.M.E.*

LAUFER, J. 1954 *N.A.C.A. Rep.* 1174.

MARSHALL, R. J. & METZNER, A. B. 1967 *Ind. Engng Chem. Fund.* **6**, 393.

MEISTER, B. J. & BIGGS, R. D. 1969 *A.I.Ch.E.J.* **15**, 643.

MERRILL, E. W., SMITH, K. A. & CHUNG, R. Y. C. 1966 *A.I.Ch.E.J.* **12**, 809.

METZNER, A. B. 1967 *A.I.Ch.E.J.* **13**, 316.

METZNER, A. B. & ASTARITA, G. 1967 *A.I.Ch.E.J.* **13**, 550.

METZNER, A. B. & METZNER, A. P. 1969 Submitted for publication.

METZNER, A. B. & PARK, M. G. 1964 *J. Fluid Mech.* **20**, 291.

METZNER A. B. & SEYER, F. A. 1966 *Proc. Sixth Naval Hydrodynamics Symposium*. Washington, D.C.: Office Naval Research.

MILLIKAN, C. B. 1939 *Proc. 5th Int. Cong. Appl. Mech.* New York: Wiley.

NICODEMO, L., ACIERNO, D. & ASTARITA, G. 1969 *Chem. Engng Sci.* **24**, 1241.

OLIVER, D. R. 1966 *Can. J. Chem. Engng*, **44**, 100.

PATTERSON, G. K., ZAKIN, J. L. & RODRIGUEZ, J. M. 1969 *Ind. Engng Chem.* **61**, 22.

RUNSTALDLER, P. W., KLINE, S. J. & REYNOLDS, W. C. 1962 *Stanford Univ. Dept. of Mech. Engng. Rep.* MD-8.

SAVINS, J. G. 1965 *A.I.Ch.E.J.* **11**, 673.

SAVINS, J. G. 1967 *Rheologica Acta*, **6**, 323.

SAVINS, J. G. 1968*a* Paper presented at *II Symposium on Viscous Drag Reduction, LTV Research Centre.*

SAVINS, J. G. 1968*b* *Rheologica Acta*, **7**, 88.

SCHLICHTING, H. 1968 *Boundary-Layer Theory*, 6th ed. New York: McGraw-Hill.

SEYER, F. A. 1968 Ph.D. Thesis, University of Delaware.

SEYER, F. A. & METZNER, A. B. 1967 *Can. J. Chem. Engng* **45**, 121.

SEYER, F. A. & METZNER, A. B. 1969 *A.I.Ch.E.J.* **15**, 426.

TOWNSEND, A. A. 1956 *The Structure of Turbulent Shear Flow*. Cambridge University Press.

TRUESDELL, C. 1964 *Phys. Fluids*, **7**, 1134.

VIRK, P. S., MERRILL, E. W., MICKLEY, H. S. & SMITH, K. A. 1967 *J. Fluid Mech.* **30**, 305.

VOGEL, W. M. & PATTERSON, A. M. 1964 *Proc. Fifth Naval Hydrodynamics Symposium*. Washington, D.C.: Office Naval Research.

VON KARMAN, T. H. 1934 *Proc. Third Int. Congr. Appl. Mechanics.*

WELLS, C. S. & SPANGLER, J. G. 1967 *Phys. Fluids*, **10**, 1890.

WETZEL, J. M. & TSAI, F. Y. 1968 *A.I.Ch.E.J.* **14**, 663.

WHITE, A. 1967 *Nature, Lond.* **214**, 585.